Exam preparation

1. The first-order tensors $\boldsymbol{a} = \boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3$ and $\boldsymbol{b} = 2\boldsymbol{e}_1 - \boldsymbol{e}_2 - \boldsymbol{e}_3$ are given with respect to an orthonormal basis $\{\boldsymbol{e}_i\}$. A second-order tensor is given by $\boldsymbol{A} = \boldsymbol{a} \otimes \boldsymbol{b}$.

   Determine:

   a) the components A_{ij} w.r.t. the basis $\{\boldsymbol{e}_i \otimes \boldsymbol{e}_j\}$,
   b) the components of $\boldsymbol{S} = \mathrm{sym}(\boldsymbol{A})$ and $\boldsymbol{W} = \mathrm{skw}(\boldsymbol{A})$ w.r.t. the basis $\{\boldsymbol{e}_i \otimes \boldsymbol{e}_j\}$,
   c) $\boldsymbol{S} \cdot \boldsymbol{W}$,
   d) $\boldsymbol{S} : \boldsymbol{W}$,
   e) $\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$

2. Determine the following operations by the aid of $\boldsymbol{\epsilon}_{\langle 3 \rangle} = \epsilon_{ijk} \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k$ and the general tensors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{A}$.

   a) $\mathrm{sym}(\boldsymbol{A}) : \mathrm{skw}(\boldsymbol{A})$,
   b) $\mathrm{tr}(\boldsymbol{a} \otimes \boldsymbol{a})^{1/2}$,
   c) $\boldsymbol{a} \cdot \boldsymbol{\epsilon}_{\langle 3 \rangle} \cdot \boldsymbol{a}$,
   d) $\mathrm{tr}(\boldsymbol{A} \cdot \boldsymbol{A}^\top)^{1/2}$,
   e) $(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{a}$

3. The vectors $\boldsymbol{v}_1 = {}^3\!/\!_5 \boldsymbol{e}_1 + {}^4\!/\!_5 \boldsymbol{e}_3$ and $\boldsymbol{v}_2 = -{}^4\!/\!_5 \boldsymbol{e}_1 + {}^3\!/\!_5 \boldsymbol{e}_3$ are given. We are in search of $\boldsymbol{v}_3$ in such a way that $\{\boldsymbol{v}_i\}$ is a (right-handed) orthonormal basis. Show whether this is possible at all. Subsequently, the product $\boldsymbol{A} \cdot \boldsymbol{v}_2$ is to be calculated where $\boldsymbol{A} = \boldsymbol{v}_1 \otimes \boldsymbol{v}_3 + \boldsymbol{v}_3 \otimes \boldsymbol{v}_1$ holds.

4. The operation $\boldsymbol{h} \times \boldsymbol{G} \times \boldsymbol{h}$ is performed with the given tensors $\boldsymbol{h} = h_i \boldsymbol{e}_i$ and $\boldsymbol{G} = G_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j$. State the components of the result in compact form with respect to the basis $\{\boldsymbol{e}_i \otimes \boldsymbol{e}_j\}$.

5. Simplify or perform the following operations with the unit tensor $\boldsymbol{I}$ and an arbitrary dyad $\boldsymbol{A}$:

   a) $\boldsymbol{\epsilon}_{\langle 3 \rangle} : \boldsymbol{I}$
   b) $\mathrm{tr}(\mathrm{skw}(\boldsymbol{A}))$
   c) $\boldsymbol{I} : (\boldsymbol{I} \otimes \boldsymbol{A}) : \boldsymbol{I}$
   d) $\boldsymbol{I} : \boldsymbol{I}$
   e) $\boldsymbol{\epsilon}_{\langle 3 \rangle} \,\vdots\, \boldsymbol{\epsilon}_{\langle 3 \rangle}$

6. Name the difference between a simple and a complete dyad.

7. Let the vectors $\boldsymbol{a} = \boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3$ and $\boldsymbol{b} = 2\boldsymbol{e}_1 - \boldsymbol{e}_2 - \boldsymbol{e}_3$ be given with respect to an orthonormal basis $\{\boldsymbol{e}_i\}$. Let $\boldsymbol{A}$ be the tensor $\boldsymbol{A} = \boldsymbol{a} \otimes \boldsymbol{b}$ and $\boldsymbol{S} = \mathrm{sym}(\boldsymbol{A})$. Determine the eigenvalues and eigenvectors of $\boldsymbol{S}$ and give the spectral decomposition.

8. Given is the Cauchy stress tensor $\boldsymbol{T} \in \mathscr{Sym}$.

   a) In many plastic processes, deformation is not dependent on pressure. A so-called stress deviator $\boldsymbol{T}'$ is therefore used: $\boldsymbol{T}' = \boldsymbol{T} - \frac{I_{\boldsymbol{T}}}{3} \boldsymbol{I}$ with $I_{\boldsymbol{T}}$ being the first principal invariant. Show that this stress measure is independent of the pressure $p := -{}^{\mathrm{tr}(\boldsymbol{T})}\!/\!_3$.
   b) Calculate the equivalent stress according to VON MISES, $T_{\mathrm{vM}} = \sqrt{3 J_2}$ with $J_2 = -I\!I_{\boldsymbol{T}'}$, with respect to the spectral representation of $\boldsymbol{T}$ (principal axis system). The three principal stresses may differ, are sorted and labeled $T_1 > T_2 > T_3$. Simplify the formula as far as possible for the spectral representation. Specify the equivalent stresses for:
      - The uni-axial tensile test with tensile stress ${}^F\!/\!_A$ and
      - the shear test, in which a block is sheared on a surface with ${}^F\!/\!_A$.

   Show that the general relationship $J_2 = \frac{1}{2} \boldsymbol{T}' : \boldsymbol{T}'$ also applies. How do you explain the factor 3 in the VON MISES formula? When is this equivalent stress suitable for evaluating a construction?

9. Let $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{d}$ be vectors. Represent them in the orthonormal basis $\boldsymbol{e}_i$ and prove the following identities:

$$\begin{aligned}
&\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) = \boldsymbol{c}\cdot(\boldsymbol{a}\times\boldsymbol{b})\,,\\
&\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = (\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}\,,\\
&(\boldsymbol{a}\times\boldsymbol{b})\times(\boldsymbol{c}\times\boldsymbol{d}) = (\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{d}))\boldsymbol{c} - (\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}))\boldsymbol{d}\,.
\end{aligned}$$

10. In the stationary case, the balance of momentum is given as follows with respect to the base $\{\boldsymbol{e}_i\}$:

$$T_{ij,j} + \rho b_i = 0$$

What is

a) the invariant tensor equation?

b) the component equation if all components are contravariant?

11. A stress tensor $\boldsymbol{T} = T_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j$ is given as follows w.r.t. an orthonormal basis:

$$[T_{ij}] = \frac{1}{2}\begin{bmatrix} T-p & 0 & -T-p \\ 0 & -2p & 0 \\ -T-p & 0 & T-p \end{bmatrix}$$

Specify the spectral representation. Is the representation obtained unique?

12. Consider the tensor $\boldsymbol{A} = \boldsymbol{a}\otimes\boldsymbol{b}$ with $\boldsymbol{a} = a_i\boldsymbol{e}_i$ and $\boldsymbol{b} = b_i\boldsymbol{e}_i$, where $\boldsymbol{a}\cdot\boldsymbol{b} = 0$ holds. Try to find a spectral representation of $\boldsymbol{A}$. Which problems arise?

13. Why and how can the product $\boldsymbol{W}\cdot\boldsymbol{x}$ with $\boldsymbol{W} \in \mathscr{Skw}$ be converted to a cross product $\boldsymbol{w}\times\boldsymbol{x}$?

14. Why are all eigenvalues of a symmetric positive definite (s. p. d.) tensor positive? Show this in the principal axis system.

15. Which tensor sets enclose the zero and the unit tensor? Tick the boxes.

$\boldsymbol{O}$	$\boldsymbol{I}$	set
☐	☐	traceless tensors
☐	☐	symmetric tensors
☐	☐	skew tensors
☐	☐	orthogonal tensors

16. Complete the empty fields in the table for second-order tensors.

tensor space	independent components	distinct eigenvalues / invariants
deviatoric ($\mathscr{Dev}$)	________	________
orthogonal ($\mathscr{Orth}$)	________	________
skew ($\mathscr{Skw}$)	________	________
spherical ($\mathscr{Sph}$)	________	________
symmetric ($\mathscr{Sym}$)	________	________

17. Let $\boldsymbol{Q} \in \mathscr{Orth}$. What can be stated about the length of $\boldsymbol{y}$ if $\boldsymbol{y} = \boldsymbol{Q}\cdot\boldsymbol{x}$?

18. In which processes is only the stress deviator $\boldsymbol{T}'$ used? Why?

19. Are tensors $\boldsymbol{K} \in \mathscr{Skw}$ invertible? Justify the answer.

20. A onefold eigenvalue λ_1 and a twofold eigenvalue λ_2 were calculated for a diagonalizable tensor $\boldsymbol{A} \in \mathscr{Lin}$. The projector for λ_1, $\boldsymbol{P}_1$ is also known. What is the projector for λ_2, $\boldsymbol{P}_2$?

21. Let the following projector representation be known for a diagonalizable tensor $\boldsymbol{A} \in \mathscr{Lin}$: $\boldsymbol{A} = \lambda_1\boldsymbol{P}_1 + \lambda_2\boldsymbol{P}_2$. State $\boldsymbol{A}^{20}$ in compact form.

22. Which equation must hold for $\boldsymbol{S} \in \mathscr{Sym}$ to be called positive definite? Can this statement also be used in general to classify $\boldsymbol{A} \in \mathscr{Lin}$? Justify your answer using the equation.
23. When can you always choose the same left and right eigenvectors in a spectral representation?
24. Show that the following invariant relations are true, using an orthonormal basis $\{\boldsymbol{e}_i\}$:

$$\mathrm{tr}(\boldsymbol{a} \otimes \boldsymbol{b}) = \boldsymbol{a} \cdot \boldsymbol{b} \qquad \text{and} \qquad \mathrm{tr}(\boldsymbol{Q} \cdot \boldsymbol{A} \cdot \boldsymbol{Q}^\top) = \mathrm{tr}(\boldsymbol{A}).$$

25. Given is a tensor $\boldsymbol{Q} \in \mathscr{Orth}$ and a vector $\boldsymbol{x} = x_i \boldsymbol{e}_i$, represented in the orthonormal basis $\{\boldsymbol{e}_i\}$. Specify the components of $\boldsymbol{y} = \boldsymbol{Q} \cdot \boldsymbol{x}$ with respect to $\{\boldsymbol{e}_i\}$ and with respect to $\{\boldsymbol{e}'_i\}$, where $\boldsymbol{Q} \cdot \boldsymbol{e}_i = \boldsymbol{e}'_i$ holds.
26. We want to rotate and reflect the orthonormal basis $\{\boldsymbol{e}_i\}$ in such a way that $\boldsymbol{e}_1 \to \boldsymbol{e}_3$, $\boldsymbol{e}_2 \to -\boldsymbol{e}_2$ and $\boldsymbol{e}_3 \to -\boldsymbol{e}_1$ holds. Specify the tensor $\boldsymbol{Q} \in \mathscr{Orth}$ that performs this, in invariant dyadic notation explicitly.
27. For a tensor $\boldsymbol{A} = A_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j$, the expression $A = \epsilon_{ijk} A_{1i} A_{2j} A_{3k}$ was calculated in an orthonormal basis $\{\boldsymbol{e}_i\}$ resulting in $A = 20$. What is the result of this expression with respect to the tangent basis $\{\boldsymbol{g}_i\}$ with $\hat{x}_1 = z^1 \cos(z^2)$, $\hat{x}^2 = z^1 \sin(z^2)$, $\hat{x}^3 = z^3$?
28. What must be true to call a basis $\{\boldsymbol{e}_i\}$ orthonormal?
29. Given are the vectors $\boldsymbol{a}_\alpha$ and $\boldsymbol{b}_\alpha$ from $\mathscr{V}$, with $\alpha \in \{1, 2, \ldots, N\}$. This forms the tensor $\boldsymbol{A}$ as follows:

$$\boldsymbol{A} = \sum_{\alpha=1}^{N} (\boldsymbol{a}_\alpha \otimes \boldsymbol{b}_\alpha - \boldsymbol{b}_\alpha \otimes \boldsymbol{a}_\alpha)$$

Determine the result of the operation $\boldsymbol{A} : \boldsymbol{1}$ and state it in compact form.

30. Tick boxes of linear functions. Let $f, x, m, n \in \mathscr{R}$, $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{f} \in \mathscr{V}$, and $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{F} \in \mathscr{Lin}$.

☐ $\boldsymbol{f}(\boldsymbol{b}) = \boldsymbol{c} \times \boldsymbol{b}$	☐ $f(\boldsymbol{A}) = \mathrm{tr}(\boldsymbol{A})$
☐ $\boldsymbol{f}(\boldsymbol{b}) = \boldsymbol{A} \cdot \boldsymbol{b}$	☐ $f(\boldsymbol{A}) = \det(\boldsymbol{A})$
☐ $\boldsymbol{f}(\boldsymbol{b}) = \boldsymbol{a} \otimes \boldsymbol{b}$	☐ $f(x) = mx + n$
☐ $\boldsymbol{F}(\boldsymbol{B}) = \boldsymbol{C} \cdot \boldsymbol{B} \cdot \boldsymbol{C}$	☐ $\boldsymbol{F}(\boldsymbol{B}) = \boldsymbol{B} \cdot \boldsymbol{C} \cdot \boldsymbol{B}$

31. Let $\boldsymbol{r}_\mathscr{O}$ be the position of a point of a body $\mathscr{B}$ with respect to the origin $\mathscr{O}$. The center of mass is $\boldsymbol{r}_\mathscr{M}$. How is the result of the integral for the decomposition $\boldsymbol{r}_\mathscr{O} = \boldsymbol{r}_\mathscr{M} + \boldsymbol{x}$:

$$\int_{\mathscr{V}} \boldsymbol{x} \, \mathrm{d}m = \ldots\ldots\ldots ?$$

Specify the single steps.

32. A stress vector $\boldsymbol{t}$ acts on the surface of a body (with normal vector $\boldsymbol{n}$). Give explicit formulae for the magnitudes of the acting normal and tangential stress.
33. Sketch examples of coordinate lines in the plane where the following applies:
    a) The tangent base is neither local nor orthogonal.
    b) The tangent base is local and orthogonal.
34. Let $\{\boldsymbol{g}_i\}$ be a tangent base and $\{\boldsymbol{g}^i\}$ the corresponding dual base. Which restriction must apply to the bases, so that subsequent formula is correct?

$$\mathrm{grad}(\boldsymbol{f}) = \frac{\partial f^i}{\partial z^j} \boldsymbol{g}_i \otimes \boldsymbol{g}^j$$

What is the generally correct version of the formula? Justify the answer.

35. Claim: The components of the unit tensor are δ_{ij}. What can be said about this?

36. Let $\boldsymbol{A} \in \mathscr{Lin}$. Is the tensor surface $\boldsymbol{x} \cdot \boldsymbol{A} \cdot \boldsymbol{x} = 1$ a suitable visualization of the complete tensor? Justify the answer.
37. How is the dyadic product defined?
38. A vector is to be decomposed using projectors. Show why the completeness of the projectors is required for this.
39. Specify the result of the operation $\exp(\vartheta \mathbf{1})$ in compact form.
40. How many invariants does a tensor in $\mathscr{R}^n$ have? Whereof are the invariants 'invariant'?
41. What does Einstein's summation convention say? In your answer, also explain the difference between tensor components with respect to orthonormal and general tangent bases.
42. Determine divergence, rotation and gradient of the vector field $\boldsymbol{f} = yz\boldsymbol{e}_x + xz\boldsymbol{e}_y + xy\boldsymbol{e}_z$.
43. The moment inertia tensor is defined as follows with respect to the center of mass:

$$\boldsymbol{J}_m = \int_{\mathscr{V}} (\boldsymbol{x}^2 \boldsymbol{I} - \boldsymbol{x} \otimes \boldsymbol{x}) \, \mathrm{d}m$$

With respect to any origin, the moment inertia tensor is defined as follows:

$$\boldsymbol{J}_{\mathscr{O}} = \int_{\mathscr{V}} (\boldsymbol{y}^2 \boldsymbol{I} - \boldsymbol{y} \otimes \boldsymbol{y}) \, \mathrm{d}m$$

Here $\boldsymbol{x}$ are vectors with respect to the center of mass m and $\boldsymbol{y}$ are vectors with respect to $\mathscr{O}$. The relationship $\boldsymbol{x} = \boldsymbol{y} - \boldsymbol{y}_m$ with a constant vector $\boldsymbol{y}_m$ with respect to $\mathscr{O}$ applies. Draw a sketch of the scenario and prove STEINER's general theorem:

$$\boldsymbol{J}_{\mathscr{O}} = \boldsymbol{J}_m + m(\boldsymbol{y}_m^2 \boldsymbol{I} - \boldsymbol{y}_m \otimes \boldsymbol{y}_m)$$

44. An astronaut encounters a spherical planet in space with a constant mass density ρ_0 and radius R without an atmosphere. A narrow channel of length $2R$ runs through this planet. The unfortunate astronaut falls into this channel at time $t = 0$. What is the astronaut's motion? After what time does he have the chance to grab the edge of the channel and get free? Assume that Newton's physics applies:

$$\boldsymbol{b}(\boldsymbol{r}) = -\mathrm{grad}(\psi(\boldsymbol{r})) \qquad \boldsymbol{\Delta}\psi(\boldsymbol{x}) = 4\pi G \rho(\boldsymbol{r})$$

The gravitational potential is bounded and unique except for one constant, furthermore it is continuous (but not necessarily the derivatives on singular surfaces). Proceed as follows:

a) Determine the operators $\boldsymbol{\Delta}$ and grad (for the application to scalar functions) in coordinates adapted to this problem. Use the symmetries of the problem and simplify the operators.

b) Compute the gravitational potential $(\boldsymbol{r})$ and the gravitational acceleration $\boldsymbol{g}(\boldsymbol{r})$ in the coordinate system adapted to the problem.

c) Set up the equation of motion and formulate the initial conditions mathematically.

d) Solve the equation of motion and answer the questions posed above.

45. What is the difference between a 3 by 3 matrix and a second-order tensor?
46. What does the statement that one tensor is perpendicular to another mean?
47. What characterizes elastic materials?
48. What properties must a general scalar product $<(\,\cdot\,), (\,\cdot\,)>$ have?
49. What is the scalar product $\langle \boldsymbol{A}_{\langle p \rangle}, \boldsymbol{B}_{\langle p \rangle} \rangle$ for tensors of p^{th} order?
50. When do you say that $\boldsymbol{A}_{\langle p \rangle}$ and $\boldsymbol{B}_{\langle p \rangle}$ are orthogonal?
51. Do you know two fourth-order projectors that are orthogonal?
52. The following dyad is given:

$$\boldsymbol{A} = \boldsymbol{e}_1 \otimes \boldsymbol{e}_2 - \boldsymbol{e}_2 \otimes \boldsymbol{e}_1$$

The basis $\{\boldsymbol{b}_i\}$ is given via the relations

$$\boldsymbol{b}_1 = \boldsymbol{e}_1, \qquad \boldsymbol{b}_2 = \boldsymbol{e}_1 + \boldsymbol{e}_2, \qquad \boldsymbol{b}_3 = \boldsymbol{e}_3 \,.$$

a) What are the properties of the tensor $\boldsymbol{A}$?

b) Specify the component matrices $\boldsymbol{A}$ and $\tilde{\boldsymbol{A}}$ so that $\boldsymbol{A} = A_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j = \tilde{A}_{ij}\boldsymbol{e}_i \otimes \boldsymbol{b}_j$ holds.

c) Is this tensor invertible? Justify the answer!

d) Which tensors are orthogonal to $\boldsymbol{A}$?

53. Given is the complete projectors group $\boldsymbol{P}_1$, $\boldsymbol{P}_2$ and $\boldsymbol{P}_3$ and the tensor $\boldsymbol{A} = 2\boldsymbol{P}_2 - \boldsymbol{P}_3$. Determine

a) $\boldsymbol{P}_1 \cdot \boldsymbol{A}$

b) $\boldsymbol{P}_2 \cdot \boldsymbol{A}$

c) $\boldsymbol{A}^3$

d) $\boldsymbol{A} - \boldsymbol{I}$. Specify the components of this result in the projector basis $\{\boldsymbol{P}_i\}$.

54. Specify whether the following formulae define permissible projectors:

a) $\boldsymbol{P}_{\text{vel}} = \boldsymbol{v} \otimes \boldsymbol{v}$, where $\boldsymbol{v}$ is the velocity

b) $\boldsymbol{P}_{\text{vel}} = \boldsymbol{n} \otimes \boldsymbol{n}$, where $\boldsymbol{n}$ is the normal vector

Give reasons for your answer.

55. Subsequent tensor functions are given:

$$\boldsymbol{A},\ \boldsymbol{b},\ \mathbb{C},\ \text{and } \mathbb{D}_{\langle 5\rangle}$$

Indicate the rank of the tensor for the results of following expressions:

a) $\mathbb{C} \otimes \operatorname{div}(\boldsymbol{A} \times \boldsymbol{b})$

b) $\operatorname{grad}(\mathbb{D}_{\langle 5\rangle} \times \operatorname{grad}(\boldsymbol{b})) : \boldsymbol{A}$

c) $\operatorname{rot}(\boldsymbol{A}) : [\mathbb{D}_{\langle 5\rangle} \vdots \mathbb{C}]$

56. Perform the operation $\boldsymbol{\epsilon}_{\langle 3\rangle} : \boldsymbol{\epsilon}_{\langle 3\rangle}$ in an orthonormal basis and state the result as compactly as possible.

57. Perform the operation $\boldsymbol{\epsilon}_{\langle 3\rangle} \vdots \boldsymbol{\epsilon}_{\langle 3\rangle}$ in an orthonormal basis and state the result as compactly as possible.

58. Determine the material time derivative of a vector field that is given a) in LAGRANGEian and b) in EULERian form.

59. The vector $\boldsymbol{c}$ is determined by the cross product.

$$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$$

Specify an invariant equivalent formula for $\boldsymbol{c}$, containing only contractions operations. The following is given for such an alternative:

$$\boldsymbol{a},\ \boldsymbol{b},\ \boldsymbol{I},\ \text{and } \boldsymbol{\epsilon}_{\langle 3\rangle}$$

60. Given is the tensor $\boldsymbol{A}$ in mixed representation.

$$\boldsymbol{A} = 2\boldsymbol{e}_1 \otimes \boldsymbol{b}_3 - 3\boldsymbol{b}_2 \otimes \boldsymbol{e}_3$$

Herein, $\{\boldsymbol{e}_i\}$ is an orthonormal basis. The following holds for the basis $\{\boldsymbol{b}_i\}$:

$$\boldsymbol{b}_1 = \boldsymbol{b}_3 \times \boldsymbol{b}_2 \qquad \boldsymbol{b}_2 = 2\boldsymbol{e}_1 \qquad \boldsymbol{b}_3 = \boldsymbol{e}_1 + \boldsymbol{e}_3$$

Determine the matrix $[A_{ij}]$, so that $\boldsymbol{A} = A_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j$ holds.

61. Given is the tensor $\boldsymbol{A}$ in mixed representation.

$$\boldsymbol{A} = 2\boldsymbol{e}_1 \otimes \boldsymbol{b}_3 - 3\boldsymbol{e}_2 \otimes \boldsymbol{b}_3 + 4\boldsymbol{e}_3 \otimes \boldsymbol{b}_1 - \boldsymbol{e}_1 \otimes \boldsymbol{b}_1 + \boldsymbol{e}_3 \otimes \boldsymbol{b}_2$$

Determine the matrix $[A_{ij}]$, so that $\boldsymbol{A} = A_{ij}\boldsymbol{e}_i \otimes \boldsymbol{b}_j$ holds. Again, $\{\boldsymbol{e}_i\}$ is an orthonormal basis. The following holds for the basis $\{\boldsymbol{b}_i\}$:

$$\boldsymbol{b}_1 = \boldsymbol{b}_3 \times \boldsymbol{b}_2 \qquad \boldsymbol{b}_2 = 2\boldsymbol{e}_1 \qquad \boldsymbol{b}_3 = \boldsymbol{e}_1 + \boldsymbol{e}_3$$

62. The following expression is given:

$$\delta_{ol}\delta_{lp}\delta_{qp}$$

a) Interpret the matrix components used as tensor components with respect to an orthonormal basis and state the equivalent invariant tensor expression.

b) Simplify the expression to the maximum, both in component and invariant notation.

63. State an invariant formula to determine the trace of the tensor $\boldsymbol{A}$. For this purpose are given:

$$\boldsymbol{A},\ \boldsymbol{A}^2,\ \boldsymbol{I},\ \boldsymbol{\epsilon}_{\langle 3\rangle}$$

64. Given a tensor $\boldsymbol{A}_{\langle p\rangle}$ with $p > 2$ and an arbitrary vector $\boldsymbol{x} \neq \mathbf{o}$. The tensor $\boldsymbol{B}_{\langle p-1\rangle}$ is determined by

$$\boldsymbol{B}_{\langle p-1\rangle} = \boldsymbol{A}_{\langle p\rangle} \cdot \boldsymbol{x}\ .$$

65. Given is a tensor $\boldsymbol{A} = A_{ij}\boldsymbol{e} \otimes \boldsymbol{e}_j$ with respect to an orthonormal basis.

$$[A_{ij}] = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

a) Determine the eigenvectors $\boldsymbol{v}_{\alpha'}$ of $\boldsymbol{A}$ with the eigenvectors given as $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$, so that $\boldsymbol{v}_{\alpha'} \cdot \boldsymbol{v}_{\beta'} = \delta_{\alpha'\beta'}$ and $\sum_{\alpha'=1}^{3} \boldsymbol{v}_{\alpha'} \otimes \boldsymbol{v}_{\alpha'} = \boldsymbol{I}$ hold.

b) What are the properties of this tensor?

c) State the inverse in the eigensystem.

d) Specify the components of the tensor $\boldsymbol{C} = \boldsymbol{A}^2 - \boldsymbol{I}$ in the eigensystem:

66. For a diagonalisable tensor $\boldsymbol{A}$, its spectral representation is known:

$$\boldsymbol{A} = \sum_{\alpha'=1}^{3} \lambda_{\alpha'}\boldsymbol{v}_{\alpha'} \otimes \boldsymbol{u}_{\alpha'}, \qquad \text{with } \boldsymbol{v}_{\alpha'} \cdot \boldsymbol{u}_{\alpha'} = \delta_{\alpha'\beta'} \qquad \text{and } \sum_{\alpha'=1}^{3} \boldsymbol{u}_{\alpha'} \otimes \boldsymbol{v}_{\alpha'} = \boldsymbol{I}$$

a) Determine the components of the spherical part of $\boldsymbol{A}$ with respect to the principal axis $\{\boldsymbol{v}_{\alpha'} \otimes \boldsymbol{u}_{\alpha'}\}$.

b) Determine the components of the deviatoric part of $\boldsymbol{A}$ with respect to the principal axis $\{\boldsymbol{v}_{\alpha'} \otimes \boldsymbol{u}_{\alpha'}\}$.

c) State a formula for the inverse of $\boldsymbol{A}$ by the aid of the spectral representation. What has to apply here?

67. Give the subsequent motion:

$$\boldsymbol{\chi}(X^1, X^2, X^3) = \alpha X^3\boldsymbol{e}_2 + \beta X^1X^2\boldsymbol{e}_1 + \kappa X^1X^3X^3\boldsymbol{e}_3 - \beta X^1X^3\boldsymbol{e}_1$$

a) State an invariant definition for the deformation gradient $\boldsymbol{F}$.

b) Determine the deformation gradient. Specify the component matrix $[F_{iJ}]$, so that $\boldsymbol{F} = F_{iJ}\boldsymbol{e}_i \otimes \boldsymbol{e}'_J$ holds.

68. Given is a tensor $\boldsymbol{A} = A_{ij}\boldsymbol{e} \otimes \boldsymbol{e}_j$ with respect to an orthonormal basis

$$[A_{ij}] = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

and the vector $\boldsymbol{x} = \boldsymbol{e}_1$. Determine the cosine of the angle enclosed by the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, defined as

$$\boldsymbol{a} = \boldsymbol{x} \cdot \boldsymbol{A} \qquad \text{and} \qquad \boldsymbol{b} = \boldsymbol{A} \cdot \boldsymbol{x}\ .$$

69. Given is the deformation gradient

$$\boldsymbol{F} = \kappa \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \cos\vartheta(\kappa^2 \boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + \kappa^3 \boldsymbol{e}_3 \otimes \boldsymbol{e}_3) + \sin\vartheta(\kappa^2 \boldsymbol{e}_3 \otimes \boldsymbol{e}_2 + \kappa^3 \boldsymbol{e}_2 \otimes \boldsymbol{e}_3)$$

with $\kappa > 0$, whereby $\boldsymbol{e}_i = \boldsymbol{e}'_I$ holds. Employ the polar decomposition of the form $\boldsymbol{F} = \boldsymbol{R} \cdot \boldsymbol{U}$ and analyze

a) (direction-dependent) stretching of the body,

b) rigid body motion, and

c) volume change.

70. State the definitions of the following operations by of the aid of the associated nabla operator $\boldsymbol{\nabla}$.

$$\operatorname{grad}(\,\cdot\,) \qquad \operatorname{Grad}(\,\cdot\,) \qquad \operatorname{div}(\,\cdot\,) \qquad \operatorname{Div}(\,\cdot\,)$$

71. Given $\boldsymbol{h} = (\boldsymbol{r} \cdot \boldsymbol{r})\boldsymbol{r}$. Determine

a) $\operatorname{grad}(\boldsymbol{h})$ and

b) $\operatorname{div}(\boldsymbol{h})$.

72. What equations define the metric components G_{ij} and G^{ij}?

73. What is the relationship between G_{ij} and G^{ij}?

74. Given is

$$[G_{ij}] = \begin{bmatrix} z^1 & 0 & 0 \\ 0 & z^2 & 0 \\ 0 & 0 & z^3 \end{bmatrix} .$$

a) What statements can be made about the corresponding coordinate lines?

b) The basis $\{\boldsymbol{g}_i\}$ is given. Give the most compact formulae possible for $\boldsymbol{g}^1$, $\boldsymbol{g}^2$ and $\boldsymbol{g}^3$.

75. Given are subsequent coordinate functions:

$$\begin{aligned} x^1 &= \hat{x}^1(z^1, z^2, z^3) := z^1 \sin z^2 \cos z^3 \\ x^2 &= \hat{x}^2(z^1, z^2, z^3) := z^1 \sin z^2 \sin z^3 \\ x^3 &= \hat{x}^3(z^1, z^2, z^3) := z^1 \cos z^2 \end{aligned}$$

Determine the tangential basis $\{\boldsymbol{g}_i\}$ and the metric components G_{ij} and G^{ij}. Simplify where possible.

76. Given are

$$[G_{ij}] = \begin{bmatrix} z^1 & 0 & 0 \\ 0 & z^2 & 0 \\ 0 & 0 & z^3 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{a}(z^1, z^2, z^3) = f(z^2)\boldsymbol{g}_1 + g(z^3)\boldsymbol{g}^2 .$$

Determine co- and contravariant components of the field $\boldsymbol{a}$.

77. How are the CHRISTOFFEL symbols Γ^i_{jk} defined?

78. Use the definitions of $\{\boldsymbol{g}_i\}$ and $\{\boldsymbol{g}^i\}$ to show any existing symmetries of the CHRISTOFFEL symbols.

79. Interprete the meaning of the CHRISTOFFEL symbol Γ^1_{23} geometrically.

80. Given is a tensor $\boldsymbol{A} \in \mathscr{Lin}$ in the form $\boldsymbol{A} = A^{ij}\boldsymbol{g}_i \otimes \boldsymbol{g}_j$. For the underlying coordinate lines, only the following CHRISTOFFEL symbols are different from zero:

$$\Gamma^1_{22} = -z^1 \qquad \Gamma^1_{12} = \Gamma^1_{21} = \frac{1}{z^1}$$

Determine $\boldsymbol{c} = \operatorname{div}(\boldsymbol{A})$ and specify the first component of the result $\boldsymbol{c}$ with respect to the base$\{\boldsymbol{g}_i\}$. Expand all sums contained.

81. Given the vector $\boldsymbol{a} = a^i \boldsymbol{g}_i$. Herein, $\{\boldsymbol{g}_i\}$ is an orthogonal tangential basis.
    a) Determine the physical components of the vector.
    b) Determine the scalar product $\boldsymbol{a} \cdot \boldsymbol{a}$ with
        i) contravariant vector components and
        ii) with physical components.

    Show the differences in calculation.
82. It was shown that tensors $\boldsymbol{A} \in \mathscr{Skw}$ are not invertible. The space $\mathscr{Skw}$ was introduced as a subset of the linear mappings between vectors in $\mathscr{R}^3$. Are skew-symmetric tensors in $\mathscr{R}^2$ also non-invertible? Answer the question using CAYLEY-HAMILTON's theorem.
83. Show that $G_{ij;k} = 0$ and $G^{ij}{}_{;k} = 0$ hold. (Recall the origin of the covariant derivative and a suitable interpretation of the metric.)
84. Two strain tensors have been introduced, $\boldsymbol{E}$ and $\boldsymbol{E}^{\mathrm{G}}$. Explain the differences in these measures. These strain tensors can be used to formulate two well known constitutive laws for elastic materials:

$$\boldsymbol{T} = \mathbb{C} : \boldsymbol{E} \qquad \text{(HOOKE)}$$

$$\boldsymbol{T} = \frac{1}{\det(\boldsymbol{F})} \boldsymbol{F} \cdot \left[\mathbb{C} : \boldsymbol{E}^{\mathrm{G}}\right] \cdot \boldsymbol{F}^{\top} \qquad \text{(ST. VENANT-KIRCHHOFF)}$$

    What are the application limits of these laws. Do they have anything in common?

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