Ableitung der Hauptinvarianten nach deren Tensoren

Hauptinvarianten von $\boldsymbol{A}$

- erste (I_A) und zweite ($I\!I_A$) Hauptinvariante

$$\begin{aligned} I_A &= \mathbf{1} \cdot \boldsymbol{A} = A_{ii} = \mathrm{sp}(\boldsymbol{A}) \\ I\!I_A &= \frac{1}{2}\left[\mathrm{sp}^2(\boldsymbol{A}) - \mathrm{sp}(\boldsymbol{A}^2)\right] \\ &= \frac{1}{2}\left[(A_{ii})^2 - (A_{ij}A_{ji})\right] \end{aligned}$$

- dritte Hauptinvariante $I\!I\!I_A$ durch I_A und $I\!I_A$ mittels des Satzes von Caley-Hamilton darstellen

$$\begin{aligned} I\!I\!I_A\, \boldsymbol{A}^0 - I\!I_A\, \boldsymbol{A}^1 + I_A\, \boldsymbol{A}^2 - \boldsymbol{A}^3 &= \mathbf{0} \\ I\!I\!I_A\, \mathbf{1} - I\!I_A\, \boldsymbol{A}^1 + I_A\, \boldsymbol{A}^2 - \boldsymbol{A}^3 &= \mathbf{0} \\ 3I\!I\!I_A - I\!I_A\, \mathrm{sp}\boldsymbol{A} + I_A\, \mathrm{sp}(\boldsymbol{A}^2) - \mathrm{sp}(\boldsymbol{A}^3) &= 0 \\ I\!I\!I_A - \frac{1}{3}I\!I_A\, \mathrm{sp}\boldsymbol{A} + \frac{1}{3}I_A\, \mathrm{sp}(\boldsymbol{A}^2) - \frac{1}{3}\mathrm{sp}(\boldsymbol{A}^3) &= 0 \end{aligned}$$

$$\begin{aligned} I\!I\!I_A &= \frac{1}{3}I\!I_A\, \mathrm{sp}\boldsymbol{A} - \frac{1}{3}I_A\, \mathrm{sp}(\boldsymbol{A}^2) + \frac{1}{3}\mathrm{sp}(\boldsymbol{A}^3) \\ I\!I\!I_A &= \frac{1}{3}I\!I_A\, A_{ii} - \frac{1}{3}I_A\, (A_{ij}A_{ji}) + \frac{1}{3}(A_{ij}A_{jm}A_{mi}) \\ I\!I\!I_A &= \frac{1}{3}\frac{1}{2}\left[(A_{ii})^2 - (A_{ij}A_{ji})\right] A_{ii} - \frac{1}{3}I_A\, (A_{ij}A_{ji}) + \frac{1}{3}(A_{ij}A_{jm}A_{mi}) \\ I\!I\!I_A &= \frac{1}{6}\left[(A_{ii})^2 - (A_{ij}A_{ji})\right] A_{ii} - \frac{1}{3}A_{ii}\, (A_{ij}A_{ji}) + \frac{1}{3}(A_{ij}A_{jm}A_{mi}) \\ I\!I\!I_A &= \frac{1}{6}(A_{ii})^3 - \frac{1}{2}A_{ii}(A_{ij}A_{ji}) + \frac{1}{3}(A_{ij}A_{jm}A_{mi}) \end{aligned}$$

Zwischenrechnung $\mathrm{sp}(\boldsymbol{A}^2)$ & $\mathrm{sp}(\boldsymbol{A}^3)$

$$\begin{aligned} \boldsymbol{A}^2 &= \boldsymbol{A}\boldsymbol{A} \\ &= A_{ij}A_{kl}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \cdot \boldsymbol{e}_k \otimes \boldsymbol{e}_l \\ &= A_{ij}A_{kl}\, \delta_{jk}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_l \\ &= A_{ij}A_{jl}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_l \\ \mathrm{sp}(\boldsymbol{A}^2) &= A_{ij}A_{ji} \end{aligned}$$

$$\begin{aligned} \boldsymbol{A}^3 &= \boldsymbol{A}\boldsymbol{A}\boldsymbol{A} \\ &= A_{ij}A_{kl}A_{mn}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \cdot \boldsymbol{e}_k \otimes \boldsymbol{e}_l \cdot \boldsymbol{e}_m \otimes \boldsymbol{e}_n \\ &= A_{ij}A_{kl}A_{mn}\, \delta_{jk}\delta_{lm}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_n \\ &= A_{ij}A_{jm}A_{mn}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_n \\ \mathrm{sp}(\boldsymbol{A}^3) &= A_{ij}A_{jm}A_{mi} \end{aligned}$$

↻

Ableitungen der Hauptinvarianten nach $\boldsymbol{A}$

$$
\begin{aligned}
\frac{\mathrm{d}I_A}{\mathrm{d}\boldsymbol{A}} &= \frac{\partial A_{ii}}{\partial A_{jk}}\,\boldsymbol{e}_j \otimes \boldsymbol{e}_k \\
&= \delta_{ij}\,\delta_{ik}\,\boldsymbol{e}_j \otimes \boldsymbol{e}_k \\
&= \delta_{jk}\,\boldsymbol{e}_j \otimes \boldsymbol{e}_k \\
&= \boldsymbol{1}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}I\!I_A}{\mathrm{d}\boldsymbol{A}} &= \frac{\frac{1}{2}\,\partial\left[(A_{ii})^2 - (A_{jk}A_{kj})\right]}{\partial A_{mn}}\,\boldsymbol{e}_m \otimes \boldsymbol{e}_n \\
&= \frac{1}{2}\left[\frac{\partial (A_{ii})^2}{\partial A_{ii}}\frac{\partial A_{ii}}{\partial A_{mn}} - \frac{\partial A_{jk}}{\partial A_{mn}}A_{kj} - A_{jk}\frac{\partial A_{kj}}{\partial A_{mn}}\right]\boldsymbol{e}_m \otimes \boldsymbol{e}_n \\
&= \frac{1}{2}\left[2A_{ii}\,\delta_{mn} - \delta_{jm}\,\delta_{kn}\,A_{kj} - A_{jk}\,\delta_{km}\,\delta_{jn}\right]\,\boldsymbol{e}_m \otimes \boldsymbol{e}_n \\
&= \frac{1}{2}\left[2\,A_{ii}\,\delta_{mn} - A_{nm} - A_{nm}\right]\,\boldsymbol{e}_m \otimes \boldsymbol{e}_n \\
&= \left[A_{ii}\,\delta_{mn} - A_{nm}\right]\,\boldsymbol{e}_m \otimes \boldsymbol{e}_n \\
&= \mathrm{sp}(\boldsymbol{A})\,\boldsymbol{1} - \boldsymbol{A}^T
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}I\!I\!I_A}{\mathrm{d}\boldsymbol{A}} &= \frac{\partial\left[\frac{1}{6}(A_{ii})^3 - \frac{1}{2}A_{ii}(A_{ij}A_{ji}) + \frac{1}{3}(A_{ij}A_{jm}A_{mi})\right]}{\partial A_{op}}\,\boldsymbol{e}_o \otimes \boldsymbol{e}_p \\
&= \left[\frac{1}{6}\frac{\partial\left[(A_{ii})^3\right]}{\partial A_{op}} - \frac{1}{2}\frac{\partial\left[A_{ii}(A_{ij}A_{ji})\right]}{\partial A_{op}} + \frac{1}{3}\frac{\partial[A_{ij}A_{jm}A_{mi}]}{\partial A_{op}}\right]\boldsymbol{e}_o \otimes \boldsymbol{e}_p \\
&= \Bigg[\frac{1}{2}(A_{ii})^2\delta_{io}\delta_{ip} - \frac{1}{2}\left(\delta_{io}\delta_{ip}(A_{ij}A_{ji}) + A_{ii}\frac{\partial\left[A_{ij}A_{ji}\right]}{\partial A_{op}}\right) \\
&\qquad + \frac{1}{3}\left(\frac{\partial A_{ij}}{\partial A_{op}}A_{jm}A_{mi} + A_{ij}\frac{\partial A_{jm}}{\partial A_{op}}A_{mi} + A_{ij}A_{jm}\frac{\partial A_{mi}}{\partial A_{op}}\right)\Bigg]\boldsymbol{e}_o \otimes \boldsymbol{e}_p \\
&= \left[\frac{1}{2}(A_{ii})^2\delta_{op} - \frac{1}{2}\delta_{op}(A_{ij}A_{ji}) - A_{ii}\boldsymbol{A}^T + \frac{1}{3}\delta_{io}\delta_{ip}A_{jm}A_{mi} + \frac{1}{3}A_{ij}\delta_{jo}\delta_{mp}A_{mi} + \frac{1}{3}A_{ij}A_{jm}\delta_{mo}\delta_{ip}\right]\boldsymbol{e}_o \otimes \boldsymbol{e}_p \\
&= \Bigg[\underbrace{\frac{1}{2}(A_{ii})^2\delta_{op} - \frac{1}{2}\delta_{op}(A_{ij}A_{ji})}_{\text{führt zu } \frac{1}{2}\,\boldsymbol{1}\,\mathrm{sp}^2(\boldsymbol{A}) - \frac{1}{2}\,\boldsymbol{1}\,\mathrm{sp}(\boldsymbol{A}^2)} \underbrace{- A_{ii}\boldsymbol{A}^T}_{\text{führt zu } I_A\boldsymbol{A}^T} \underbrace{+ \frac{1}{3}\delta_{op}A_{jm}A_{mi} + \frac{1}{3}A_{ij}\delta_{op}A_{mi} + \frac{1}{3}A_{ij}A_{jm}\delta_{op}}_{\text{führt zu } \boldsymbol{A}^{2T}}\Bigg]\boldsymbol{e}_o \otimes \boldsymbol{e}_p \\
&= \underbrace{\frac{1}{2}\,\boldsymbol{1}\,\mathrm{sp}^2(\boldsymbol{A}) - \frac{1}{2}\,\boldsymbol{1}\,\mathrm{sp}(\boldsymbol{A}^2)}_{I\!I_A\boldsymbol{1}} \qquad - I_A\boldsymbol{A}^T \qquad + \boldsymbol{A}^{2T} \\
&= I\!I_A\boldsymbol{1} - I_A\boldsymbol{A}^T + \boldsymbol{A}^{2T}
\end{aligned}
$$