Dualbasen & Metrikkoeffizienten

Geg.:

$$\boldsymbol{g}_1 = \boldsymbol{e}_1 + 3\,\boldsymbol{e}_2$$
$$\boldsymbol{g}_2 = 2\,\boldsymbol{e}_1$$

Ges.:

$G_{ij}, \boldsymbol{g}^i, G^{ij}$ und Probe $G_{ij}\,G^{jk} = \delta_{jk}$

Lösung: $G_{ij} = \boldsymbol{g}_i \cdot \boldsymbol{g}_j$

$$G_{11} = \boldsymbol{g}_1 \cdot \boldsymbol{g}_1 = (\boldsymbol{e}_1 + 3\,\boldsymbol{e}_2)\cdot(\boldsymbol{e}_1 + 3\,\boldsymbol{e}_2) = \underbrace{\boldsymbol{e}_1 \cdot \boldsymbol{e}_1}_{1} + 3\underbrace{\boldsymbol{e}_2 \cdot \boldsymbol{e}_1}_{0} + 3\underbrace{\boldsymbol{e}_2 \cdot \boldsymbol{e}_1}_{0} + 9\underbrace{\boldsymbol{e}_2 \cdot \boldsymbol{e}_2}_{1} = 10$$

$$G_{12} = \boldsymbol{g}_1 \cdot \boldsymbol{g}_2 = (\boldsymbol{e}_1 + 3\,\boldsymbol{e}_2)\cdot(2\,\boldsymbol{e}_1) = 2\underbrace{\boldsymbol{e}_1 \cdot \boldsymbol{e}_1}_{1} + 6\underbrace{\boldsymbol{e}_2 \cdot \boldsymbol{e}_1}_{0} = 2$$

$$G_{21} = \boldsymbol{g}_2 \cdot \boldsymbol{g}_1 = (2\,\boldsymbol{e}_1)\cdot(\boldsymbol{e}_1 + 3\,\boldsymbol{e}_2) = 2\underbrace{\boldsymbol{e}_1 \cdot \boldsymbol{e}_1}_{1} + 6\underbrace{\boldsymbol{e}_1 \cdot \boldsymbol{e}_2}_{0} = 2$$

$$G_{22} = \boldsymbol{g}_2 \cdot \boldsymbol{g}_2 = (2\,\boldsymbol{e}_1)\cdot(2\,\boldsymbol{e}_1) = 4\underbrace{\boldsymbol{e}_1 \cdot \boldsymbol{e}_1}_{1} = 4$$

$$\Rightarrow G_{ij} = \begin{bmatrix} 10 & 2 \\ 2 & 4 \end{bmatrix}$$

Lösung: $\boldsymbol{g}^i$

$$\boldsymbol{g}^1 = a\,\boldsymbol{e}_1 + b\,\boldsymbol{e}_2$$
$$\boldsymbol{g}^2 = c\,\boldsymbol{e}_1 + d\,\boldsymbol{e}_2$$

$$\boldsymbol{g}_1 \cdot \boldsymbol{g}^1 = 1 = (\boldsymbol{e}_1 + 3\,\boldsymbol{e}_2)\cdot(a\,\boldsymbol{e}_1 + b\,\boldsymbol{e}_2) = a\underbrace{\boldsymbol{e}_1 \cdot \boldsymbol{e}_1}_{1} + b\underbrace{\boldsymbol{e}_1 \cdot \boldsymbol{e}_2}_{0} + 3a\underbrace{\boldsymbol{e}_2 \cdot \boldsymbol{e}_1}_{0} + 3b\underbrace{\boldsymbol{e}_2 \cdot \boldsymbol{e}_2}_{1} = a + 3b$$

$$\Rightarrow b = \frac{1}{3} - \frac{1}{3}a = \frac{1}{3}$$

$$\boldsymbol{g}_1 \cdot \boldsymbol{g}^2 = 0 = (\boldsymbol{e}_1 + 3\,\boldsymbol{e}_2)\cdot(c\,\boldsymbol{e}_1 + d\,\boldsymbol{e}_2) = c\underbrace{\boldsymbol{e}_1 \cdot \boldsymbol{e}_1}_{1} + d\underbrace{\boldsymbol{e}_1 \cdot \boldsymbol{e}_2}_{0} + 3c\underbrace{\boldsymbol{e}_2 \cdot \boldsymbol{e}_1}_{0} + 3d\underbrace{\boldsymbol{e}_2 \cdot \boldsymbol{e}_2}_{1} = c + 3d$$

$$\Rightarrow d = -\frac{1}{3}c = -\frac{1}{6}$$

$$\boldsymbol{g}_2 \cdot \boldsymbol{g}^1 = 0 = (2\,\boldsymbol{e}_1)\cdot(a\,\boldsymbol{e}_1 + b\,\boldsymbol{e}_2) = 2a\underbrace{\boldsymbol{e}_1 \cdot \boldsymbol{e}_1}_{1} + 2b\underbrace{\boldsymbol{e}_1 \cdot \boldsymbol{e}_2}_{0} = 2a$$

$$\Rightarrow a = 0$$

$$\boldsymbol{g}_2 \cdot \boldsymbol{g}^2 = 1 = (2\,\boldsymbol{e}_1)\cdot(c\,\boldsymbol{e}_1 + d\,\boldsymbol{e}_2) = 2c\underbrace{\boldsymbol{e}_1 \cdot \boldsymbol{e}_1}_{1} + 2d\underbrace{\boldsymbol{e}_1 \cdot \boldsymbol{e}_2}_{0} = 2c$$

$$\Rightarrow c = \frac{1}{2}$$

also:

$$\boldsymbol{g}^1 = a\,\boldsymbol{e}_1 + b\,\boldsymbol{e}_2 \qquad = 0\,\boldsymbol{e}_1 + \left(\frac{1}{3}\right)\,\boldsymbol{e}_2 \qquad = \frac{1}{3}\,\boldsymbol{e}_2$$

$$\boldsymbol{g}^2 = c\,\boldsymbol{e}_1 + d\,\boldsymbol{e}_2 \qquad = \frac{1}{2}\,\boldsymbol{e}_1 + \left(-\frac{1}{6}\right)\,\boldsymbol{e}_2 \qquad = \frac{1}{2}\,\boldsymbol{e}_1 - \frac{1}{6}\,\boldsymbol{e}_2$$

Lösung $G^{ij} = \boldsymbol{g}^i \cdot \boldsymbol{g}^j$

$$G^{11} = \boldsymbol{g}^1 \cdot \boldsymbol{g}^1 = \left(\frac{1}{3}\,\boldsymbol{e}_2\right) \cdot \left(\frac{1}{3}\,\boldsymbol{e}_2\right) \qquad = \frac{1}{9}\,\underbrace{\boldsymbol{e}_1 \cdot \boldsymbol{e}_1}_{1} = +\frac{1}{9}$$

$$G^{12} = \boldsymbol{g}^1 \cdot \boldsymbol{g}^2 = \left(\frac{1}{3}\,\boldsymbol{e}_2\right) \cdot \left(\frac{1}{2}\,\boldsymbol{e}_1 - \frac{1}{6}\,\boldsymbol{e}_2\right) \qquad = \frac{1}{6}\,\underbrace{\boldsymbol{e}_2 \cdot \boldsymbol{e}_1}_{0} - \frac{1}{18}\,\underbrace{\boldsymbol{e}_2 \cdot \boldsymbol{e}_2}_{1} = -\frac{1}{18}$$

$$G^{21} = \boldsymbol{g}^2 \cdot \boldsymbol{g}^1 = \left(\frac{1}{2}\,\boldsymbol{e}_1 - \frac{1}{6}\,\boldsymbol{e}_2\right) \cdot \left(\frac{1}{3}\,\boldsymbol{e}_2\right) \qquad = \frac{1}{6}\,\underbrace{\boldsymbol{e}_1 \cdot \boldsymbol{e}_2}_{0} - \frac{1}{18}\,\underbrace{\boldsymbol{e}_2 \cdot \boldsymbol{e}_2}_{1} = -\frac{1}{18}$$

$$G^{22} = \boldsymbol{g}^2 \cdot \boldsymbol{g}^2 = \left(\frac{1}{2}\,\boldsymbol{e}_1 - \frac{1}{6}\,\boldsymbol{e}_2\right) \cdot \left(\frac{1}{2}\,\boldsymbol{e}_1 - \frac{1}{6}\,\boldsymbol{e}_2\right) \quad = \frac{1}{4}\,\underbrace{\boldsymbol{e}_1 \cdot \boldsymbol{e}_1}_{1} - \frac{1}{12}\,\underbrace{\boldsymbol{e}_1 \cdot \boldsymbol{e}_2}_{0} - \frac{1}{12}\,\underbrace{\boldsymbol{e}_2 \cdot \boldsymbol{e}_1}_{0} + \frac{1}{36}\,\underbrace{\boldsymbol{e}_2 \cdot \boldsymbol{e}_2}_{1} = +\frac{5}{18}$$

$$\Rightarrow G^{ij} = \frac{1}{18}\begin{bmatrix} 2 & -1 \\ -1 & 5 \end{bmatrix}$$

Test:

$$G^{ij} = (G_{ij})^{-1} = \begin{bmatrix} 10 & 2 \\ 2 & 4 \end{bmatrix}^{-1} = \frac{1}{\det(G_{ij})}\begin{bmatrix} 4 & -2 \\ -2 & 10 \end{bmatrix} = \frac{1}{10 \cdot 4 - (-2 \cdot (-2))}\begin{bmatrix} 4 & -2 \\ -2 & 10 \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 2 & -1 \\ -1 & 5 \end{bmatrix}$$

Probe: $G_{ij}G^{jk} = \delta_{jk}$

$$G_{ij}G^{jk} = \begin{bmatrix} 10 & 2 \\ 2 & 4 \end{bmatrix} \cdot \begin{bmatrix} 2/18 & -1/18 \\ -1/18 & 5/18 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \delta_{jk}$$

Skizze: