Exercise 05

1. Projector representation of isotropic fourth-order tensor

Problem

- A projector decomposition of an arbitrary symmetric tensor of fourth-order is of the form

$$\mathbb{C} = \sum_{\alpha=1}^{N} \lambda_\alpha \mathbb{P}_\alpha \,.$$

Let N be the number of distinct eigenvalues λ_α of $\mathbb{C}$. Then the projectors $\mathbb{P}_\alpha$ fulfill subsequent projector rules:

idempotence: $\mathbb{P}_\alpha \mathbb{P}_\alpha = \mathbb{P}_\alpha \qquad \forall\, \alpha = \{1, \ldots, N\}$

orthogonality: $\mathbb{P}_\alpha \mathbb{P}_\beta = \mathbb{O} \qquad \forall\, \alpha \neq \beta$

completeness: $\sum \mathbb{P}_\alpha = \mathbb{I}$

If there are only simple eigenvalues, the projector decomposition coincides with the spectral form of the tensor. Determine the projector decomposition of an isotropic $\mathbb{C}$.

Solution

- The fourth-order identity tensor is given with respect to an ortonormal basis $\{\boldsymbol{e}_i\}$.

$$\mathbb{I} = \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_i \otimes \boldsymbol{e}_j$$

- The fourth-order transposer is given as follows.

$$\mathbb{T} = \boldsymbol{e}_j \otimes \boldsymbol{e}_i \otimes \boldsymbol{e}_i \otimes \boldsymbol{e}_j$$

- The fourth-order symmetrizer is given as

$$\mathbb{I}^{\text{sym}} = \frac{1}{2}\left(\mathbb{I} + \mathbb{T}\right)$$

while the fourth-order anti(sym)metrizer is

$$\mathbb{I}^{\text{skw}} = \frac{1}{2}\left(\mathbb{I} - \mathbb{T}\right) \,.$$

The isotropic fourth-order tensor is given, while we additionally apply above introduced abbreviations

$$\begin{aligned}
\mathbb{C} &= a\, \boldsymbol{e}_i \otimes \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_j + b\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_i \otimes \boldsymbol{e}_j + c\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_i \\
&= a\, \boldsymbol{I} \otimes \boldsymbol{I} \qquad + (b+c)\, \mathbb{I}^{\text{sym}} \qquad + (b-c)\, \mathbb{I}^{\text{skw}} \\
&= (3a+b+c)\mathbb{P}_1 \qquad + (b+c)\mathbb{P}_2 \qquad + (b-c)\mathbb{P}_3
\end{aligned}$$

with $a, b, c \in \mathscr{R}$ (set of real numbers) and

$$\mathbb{P}_1 = \frac{1}{3}\boldsymbol{I} \otimes \boldsymbol{I} \qquad \mathbb{P}_2 = \mathbb{I}^{\text{sym}} - \mathbb{P}_1 \qquad \mathbb{P}_3 = \mathbb{I}^{\text{skw}} \,.$$

These tensors fulfill the projector rules. Thus, the eigenvalues of $\mathbb{C}$ are as follows.

$$\lambda_1 = 3a + b + c \qquad \lambda_2 = b + c \qquad \lambda_3 = b - c$$

2. Value ranges of material parameters

Problem

- Determine the constraints on the value range for Young's modulus and Poisson's ratio, which arise from the requirement of positive definiteness of the stiffness tetrad of the isotropic St. Venant-Kirchhoff material law

$$\mathbb{C} = \lambda \boldsymbol{I} \otimes \boldsymbol{I} + 2\mu \mathbb{I}^{\mathrm{sym}}$$

with λ and μ beeing Lamé parameters.

Solution

- The spectral representation of this law is given as

$$\mathbb{C} = 3K\mathbb{P}_1 + 2G\mathbb{P}_2$$

with bulk modulus $K = \lambda + {}^2\!/_3\mu$ and shear modulus $G = \mu$. $\mathbb{P}_\alpha$ are eigenspace projectors:

$$\mathbb{P}_1 = \frac{1}{3}\boldsymbol{I} \otimes \boldsymbol{I} \qquad \mathbb{P}_2 = \mathbb{I}^{\mathrm{sym}} - \mathbb{P}_1$$

- The projector rules are as follows:

$$\begin{aligned} &\text{idempotence:} && \mathbb{P}_\alpha\mathbb{P}_\alpha = \mathbb{P}_\alpha && \forall\, \alpha = \{1, 2\} \\ &\text{orthogonality:} && \mathbb{P}_1\mathbb{P}_2 = \mathbb{P}_2\mathbb{P}_1 = \mathbb{O} \\ &\text{completeness:} && \mathbb{P}_1 + \mathbb{P}_2 = \mathbb{I}^{\mathrm{sym}} \end{aligned}$$

- The projector representation implies that $3K$ and $2G$ are the onefold and the fivefold eigenvalue of $\mathbb{C}$. Since $\mathbb{C}$ is (major) symmetric and positive definite, these eigenvalues are also positive.

$$K > 0 \qquad G > 0$$

- The relation between K and G as well as Young's modulus Y and Poisson's ratio ν is analogous to the Hooke's law.

$$3K = \frac{Y}{1 - 2\nu} \qquad 2G = \frac{Y}{1 + \nu}$$

- From $K > 0$ and $G > 0$ follows

$$0 < \frac{1}{9K} + \frac{1}{3G} = \frac{1}{Y}\ .$$

This indicates the $Y > 0$ for Young's modulus. We can transform the relations from two equations above

$$\nu_- = \frac{1}{2}\frac{Y}{G} - 1 \qquad \nu_+ = -\frac{1}{6}\frac{Y}{K} + \frac{1}{2}$$

and set the bounds ${}^Y\!/_G, {}^Y\!/_K \to 0$ to determine the range of Poisson's ratio: $-1 < \nu < {}^1\!/_2$