Exercise 02

1. Inversion of the Rayleigh product

Problem

- Let $\boldsymbol{A}$ be an invertible second-order tensor and $\mathbb{C}$ an invertible fourth-order tensor. Prove

$$(\boldsymbol{Q} \star \mathbb{C})^{-1} = \boldsymbol{A}^{-\top} \star \mathbb{C}^{-1} \,.$$

Solution

- We show this by

$$(\boldsymbol{Q} \star \mathbb{C})^{-1} (\boldsymbol{Q} \star \mathbb{C}) = \mathbb{I} \,.$$

- In fact,

$$\begin{aligned}
\left(\boldsymbol{A}^{-\top} \star \mathbb{C}^{-1}\right)_{ijmn} (\boldsymbol{Q} \star \mathbb{C})_{mnkl} &= A^{-\top}_{ia} A^{-\top}_{jb} A^{-\top}_{mc} A^{-\top}_{nd} C^{-1}_{abcd} \; A_{me} A_{nf} A_{kg} A_{lh} C_{lfgh} \\
&= A^{-1}_{ai} A^{-1}_{bj} A^{-1}_{cm} A^{-1}_{dn} C^{-1}_{abcd} \; A_{me} A_{nf} A_{kg} A_{lh} C_{lfgh} \\
&= A^{-1}_{ai} A^{-1}_{bj} \; \delta_{ce} \delta_{df} \; A_{kg} A_{lh} \; C^{-1}_{abcd} C_{efgh} \\
&= A^{-1}_{ai} A^{-1}_{bj} \; A_{kg} A_{lh} \; C^{-1}_{abef} C_{efgh} \\
&= A^{-1}_{ai} A^{-1}_{bj} \; A_{kg} A_{lh} \; \delta_{ag} \delta_{bh} \\
&= A^{-1}_{gi} A^{-1}_{hj} \; A_{kg} A_{lh} \\
&= \delta_{ik} \delta_{jl} \,.
\end{aligned}$$

These are the components of the identity tensor $\mathbb{I} = \delta_{ik}\delta_{jl} \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l$. Therefore, $\boldsymbol{A}^{-\top} \star \mathbb{C}^{-1}$ is the inverse of $\boldsymbol{Q} \star \mathbb{C}$.

2. Invariance of isotropic fourth-order tensor

Problem

- The following tensor is called isotropic fourth-order tensor

$$\mathbb{C} = a \, \boldsymbol{e}_i \otimes \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_j + b \, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_i \otimes \boldsymbol{e}_j + c \, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_i$$

where $a, b, c \in \mathscr{R}$ holds and $\{\boldsymbol{e}_i\}$ is an orthonormal basis. Show that $\mathbb{C}$ is invariant under the Rayleigh product $\boldsymbol{Q} \star \mathbb{C} = \mathbb{C}$ for all $\boldsymbol{Q} \in \mathscr{O}\mathrm{rth}$.

Solution

- The components of $\mathbb{C}$ with respect to the orthonormal basis are

$$C_{ijkl} = a \, \delta_{ij} \delta_{kl} + b \, \delta_{ik} \delta_{jl} + c \, \delta_{il} \delta_{jk} \,.$$

- The components of $\boldsymbol{Q} \star \mathbb{C}$ with respect to the same orthonormal basis are

$$\begin{aligned}
(\boldsymbol{Q} \star \mathbb{C})_{mnop} &= Q_{mi} Q_{nj} Q_{ok} Q_{pl} C_{ijkl} \\
&= a \, Q_{mi} Q_{ni} Q_{ok} Q_{pk} + b \, Q_{mi} Q_{nj} Q_{oi} Q_{pj} + c \, Q_{mi} Q_{nj} Q_{oj} Q_{pi} \\
&= a \, \delta_{mn} \delta_{op} + b \, \delta_{mo} \delta_{np} + c \, \delta_{mp} \delta_{no} \,.
\end{aligned}$$

The direct juxtaposition reveals the equality

$$\begin{aligned}
C_{ijkl} =& a \, \delta_{ij} \;\; \delta_{kl} + b \, \delta_{ik} \;\; \delta_{jl} \;\; + c \, \delta_{il} \;\; \delta_{jk} \\
=& a \, \delta_{mn} \delta_{op} + b \, \delta_{mo} \delta_{np} + c \, \delta_{mp} \delta_{no}
\end{aligned}$$

so that the invariance is proved.